- Sem - Linear homogenization

- Following: homogenization of differential operators & integral functionals by
  | Jikov
  | Kozlov
  | Olienik

- First results by Kozlov, Papanicolaou, Varadhan, '79
- We are interested in

$$\operatorname{div}\left( A\left(\tfrac{x}{\varepsilon}, \omega\right) \nabla u^{\varepsilon}(x,\omega) \right) = f(x)$$

$x \in \mathbb{R}^n$, $\omega \in \Omega$, $\Omega$ probability space.
$\omega$: random coefficients

- goal: $u^{\varepsilon} \longrightarrow u^{0}$ in some sense, almost surely, where $u^0$ satisfies an homogenized PDE

$$\operatorname{div}( A^0 \nabla u^0) = f$$

- Let $A: \mathbb{R}^d \times \Omega \longrightarrow \mathbb{R}^{d\times d}_{sym}$ measurable s.t. $\exists\ \nu_1, \nu_2 > 0$ with

$$\nu_1 |\xi|^2 \leq A(x,\omega)\xi \cdot \xi \leq \nu_2 |\xi|^2$$

$\forall \xi, x \in \mathbb{R}^d$, a.s. in $\Omega$.

The important assumption is stationarity:

$$\left( A(\cdot, x_1 + h), \dots, A(\cdot, x_n + h) \right) \text{ in } \left( \mathbb{R}^{d\times d} \right)^n$$

is invariant under changes in h. $\forall n \in \mathbb{N}$ $\forall h \in \mathbb{R}^d$

- It's useful to instead consider the law of $A$, $\mu$ on
  $\mathcal{E}(\nu_1, \nu_2) := \{ A \in \mathbb{R}^{d\times d} \text{ if } \nu_1 |\xi|^2 \le A\xi\cdot\xi \le \nu_2 |\xi|^2 \}$

- Def.: a d-dimensional dynamical system on $\Omega$ is a family $T_x : \Omega \to \Omega$, $x \in \mathbb{R}^d$ s.t.

  i) $T_0 = Id$, $T_{x+y} = T_x T_y$

  ii) $T_x$ are measure preserving [i.e., $\mu(T_x(E)) = \mu(E)$ $\forall E \subset \Omega$ measurable]

  this is the stationary assumption

  iii) $(x, \omega) \mapsto f(T_x \omega)$ is jointly measurable on $\Omega \times \mathbb{R}^d$ $\forall f$ measurable on $\Omega$.

## Ergodicity:

- **Def:** For a dynamical system $T_x$ on a probability space $(\Omega, \mathcal{F}, \mu)$, we say that it is **ergodic** if one of the following two equivalent properties holds:

  i) $\exists x \neq 0$ s.t. $T_x E = E \Rightarrow \mu(E) \in \{0,1\}$

  ii) $\exists x$, $f$ measurable s.t.

  $$f(T_x \omega) = f(\omega) \text{ a.s.} \Rightarrow f \text{ constant a.s.}$$

we don't have small parts of our space that are mapped into themselves

- **Def:** Let $f \in L^1_{loc}(\mathbb{R}^d)$, we say that $f$ has a mean value $M(f)$ if $\forall K \subseteq \mathbb{R}^d$ measurable & bounded, it holds

$$\lim_{\varepsilon \to \infty} \frac{1}{\varepsilon^d |K|} \int_{\varepsilon K} f(x)\,dx = \lim_{\varepsilon \to 0} \fint_K f\left(\frac{x}{\varepsilon}\right) dx = M(f).$$

- Def.: Let $Inv^2(\Omega)$ be the subspace of $L^2(\Omega)$ s.t. $f(T_x\omega) = f(\omega)$ a.s. $\forall x \in \mathbb{R}^d$.

- Ergodic thm: (pointwise)

  Let $f \in L^2(\Omega)$. Consider the function $f(x,\omega) := f(T_x\omega)$. Then $f$ has a mean value a.s., and $M(f(T_\cdot\omega))$ is the $L^2$-projection of $f$ into $Inv^2(\Omega)$. Moreover, if $\mu$ is ergodic, then

$$M(f(T_\cdot\omega)) = \mathbb{E}[f] = \langle f \rangle.$$

- Note: this is equivalent to

$$f(T_{\frac{x}{\varepsilon}}\omega) \xrightarrow{w\text{-}L^2_{loc}} M(f(T_x\omega)) \quad \text{a.s.}$$